

# Adaptive Fractal-like Network Structure for Efficient Search of Inhomogeneously Distributed Targets at Unknown Positions<sup>1</sup>

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**Abstract**—Since a spatial distribution of communication requests is inhomogeneous and related to a population, in constructing a network, it is crucial for delivering packets on short paths through the links between proximity nodes and for distributing the load of nodes how to locate the nodes as base-stations on a realistic wireless environment. In this paper, from viewpoints of complex network science and biological foraging, we propose a scalably self-organized geographical network, in which the proper positions of nodes and the network topology are simultaneously determined according to the population, by iterative divisions of rectangles for load balancing of nodes in the adaptive change of their territories. In particular, we consider a decentralized routing by using only local information, and show that, for searching targets around high population areas, the routing on the naturally embedded fractal-like structure by population has higher efficiency than the conventionally optimal strategy on a square lattice.

**Keywords**—self-organised design; wide-area wireless communication; routing in ad hoc networks; random walk; Levy flight

## I. INTRODUCTION

Many network infrastructures: power grids, airline networks, and the Internet, are embedded in a metric space, and long-range links are relatively restricted [1], [2] for economical reasons. The spatial distribution of nodes is neither uniformly at random nor on a regular lattice, which is often assumed in the conventional network models. In real data, a population density is mapped to the number of router nodes on Earth [1]. Similar spatially inhomogeneous distributions of nodes are found in air transportation networks [3] and in mobile communication networks [4]. Thus, it is not trivial how to locate nodes on a space in a pattern formation of points. Point processes in spatial statistics [5] provide models for irregular patterns of points in urban planning, astronomy, forestry, or ecology, such as spatial distributions of rainfall, germinations, plants, and animals. The processes assume homogeneous Poisson and Gibbs distributions to generate a pattern of random packing or independent clustering, and to estimate parameters of competitive potential functions in a territory model for a given statistical data, respectively. However, rather than random pattern and statistical estimation, **we focus on a self-organized network infrastructure by taking**

**into account realistic spatial distributions of nodes and communication requests.** In particular, we aim to develop adaptive and scalable ad hoc networks by adding the links between proximity nodes according to the increasing of communication requests. Because a spatial distribution of communication requests affect the proper positions of nodes, which control both the load of requests assigned to each node (e.g., assigned at the nearest access point of node as a base-station from a user) and the communication efficiency depending on the selection of routing paths.

For a routing in ad hoc networks, global information, e.g., a routing table in the Internet, cannot be applied, because many nodes and connections between them are likely to change over time. In early works on computer science, some decentralized routing methods were developed to reduce energy consumption in sensor or mobile networks. However they lead to the failure of guaranteed delivery [6]; in the flooding algorithm, multiple redundant copies of a message are sent and cause congestion, while greedy and compass routings may occasionally fall into infinite loops or into a dead end. In complex network science, other efficient decentralized routing methods have been also proposed. The stochastic methods by using local information of the node degrees and other measures are called preferential [7] and congestion-aware [8], [9] routings as extensions of a uniformly random walk.

Decentralized routing has a potential performance to search a target whose position is unknown in advance. Since this situation looks like foraging, the biological strategy may be useful for the efficient search. **We are interested in a relation of the search and the routing on a spatially inhomogeneous network structure according to a population.** Many experimental observations for biological foraging found the evidence in favor of anomalous diffusion in the movement of insects, fishes, birds, mammals and human being [10]. As the consistent result, it has been theoretically analyzed for a continuous space model that an inverse square root distribution of flight lengths is an optimal strategy to search sparsely and randomly located targets on a homogeneous space [11]. The discrete space models on a regular lattice [12] and the defective one [13] are also discussed. Such behavior is called *Levy flight* characterized by a distribution function  $P(l_{ij}) \sim l_{ij}^{-\mu}$  with  $1 < \mu \leq 3$ , where  $l_{ij}$  is a flight length between nodes  $i$

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and  $j$  in the stochastic movement for any direction. The values of  $\mu \geq 3$  lead to Brownian motions, while  $\mu \rightarrow 1$  to ballistic motions. The optimal case is  $\mu \approx 2$  for maximizing the efficiency of search. Here, we assume that the mobility of a node is ignored due to a sufficiently slow speed in comparison with the communication process. In the current or future technologies, wide-area wireless connections by directional beams will be possible, the modeling of unit disk graph with a constant transmission range is not necessary. Thus, **we propose a scalably self-organized geographical network and show that the naturally embedded fractal-like structure is suitable for searching targets more efficiently than the square lattice tracked by the Levy flights.**

## II. GEOGRAPHICAL NETWORKS

### A. Conventional Models

Geographical constructions of complex networks have been proposed so far. As a typical generation mechanism of scale-free (SF) networks that follow a power law degree distribution found in many real systems [14], [15], a spatially preferential attachment is applied in some extensions [16], [17], [18], [19], [20] from the topological degree based model [21] to a combination of degree and distance based model. The original preferential attachment is known as “rich gets richer” rule that means a higher degree node tend to get more links. On the other hand, geometric construction methods have also been proposed. They have both small-world [22] and SF structures generated by a recursive growing rule for the division of a chosen triangle [23], [24], [25], [26] or for the attachment aiming at a chosen edge [27], [28], [29] in random or hierarchical selections. These models are proper for the analysis of degree distribution due to the regularly recursive generation process. Although the position of a newly added node is basically free as far as the geometric operations are possible, it has no relation to population. Considering the effects of population on a geographical network is necessary to self-organize a spatial distribution of nodes that is suitable for socioeconomic communication and transportation requests. Moreover, in these geometric methods, narrow triangles with long links tend to be constructed, and adding only one node per step may lead to exclude other topologies from the SF structure. Unfortunately, SF networks are extremely vulnerable against the intentional hub attacks [30]. We should develop other self-organizations of network apart from the conventional models; e.g. a better network without long links can be constructed by subdivisions of equilateral triangle which is a well balanced (neither fat nor thin) shape for any directions.

### B. Generalized Multi-Scale Quartered Network

Thus, we have considered the multi-scale quartered (MSQ) network model [31], [32]. It is based on a stochastic construction by a self-similar tiling of primitive shape, such

as an equilateral triangle or square. The MSQ networks have several advantages of the strong robustness of connectivity against node removals by random failures and intentional attacks, the bounded short path as  $t = 2$ -spanner [33], and the efficient face routing by using only local information. Furthermore, the MSQ networks are more efficient (economic) with shorter link lengths and more suitable (tolerant) with lower load for avoiding traffic congestion [32] than the state-of-the-art geometric growing networks [23], [27], [28], [29], [24], [25], [26] and the spatially preferential attachment models [16], [17], [18], [19], [20] with various topologies ranging from river to SF geographical networks. However, in the MSQ networks, the position of a new node is restricted on the half-point of an edge of the chosen face, and the link length is proportional to  $(\frac{1}{2})^H$  where  $H$  is the depth number of iterative divisions. Thus, from square to rectangle, we generalize the division procedures as follows.

Step0: Set an initial square whose inside are the candidates of division axes as the segments of a  $L \times L$  lattice.

Step1: At each time step, a face is chosen with a probability proportional to the population counted in the face covered by mesh blocks of a census data.

Step2: Four smaller rectangles are created from the division of the chosen rectangle face by horizontal and vertical axes. For the division, two axes are chosen by that their cross point is the nearest to the population barycenter of the face.

Step3: Return to Step 1, while the network size (the total number of nodes)  $N$  does not exceed a given size.

Note that the maximum size  $N_{max}$  depends on the value of  $L$ ; the iteration of division is finitely stopped, since the extreme rectangle can not be divided any longer when one of the edge lengths of rectangle is the initial lattice’s unit length. We use the population data on a map in  $80km^2$  of  $160 \times 160$  mesh blocks ( $L = 160$ ) provided by the Japan Statistical Association. Of course, other data is possible.

It is worth noting that the positions of nodes and the network topology are simultaneously determined by the divisions of faces within the fractal-like structure. There exists a mixture of sparse and dense parts of nodes with small and large faces. Moreover, with the growing network, the divisions of faces perform a load balancing of nodes in their adaptively changed territories for the population. We emphasize that such a network is constructed according to a spatially inhomogeneous distribution of population which is proportional to communication requests in a realistic environment. In the following, we show the naturally embedded fractal-like structure are suitable for searching targets.

## III. SEARCH PERFORMANCE

As a preliminary, we consider the preferential routing [7] which is also called  $\alpha$ -random walk [34]; The forwarding node  $j$  is chosen proportionally to  $K_j^\alpha$  by a walker in the

connected one hop neighbors  $\mathcal{N}_i$  of its resident node  $i$  of a walker (packet), where  $K_j$  denotes the degree of node  $j$  and  $\alpha$  is a real parameter. We assume that the start position of walker is set to the nearest node to the population barycenter of the initial square. Figure 1 shows the length distribution of visited links. The dashed lines in log-log plot imply a power law, for which the exponents estimated as the slopes by a mean-square-error method are 2.336, 2.315, and 2.296 for  $\alpha = 1, 0, -1$ , respectively. These values are close to the optimal exponent  $\mu \approx 2$  [11], [12] in the Levy flight on a square lattice. The exponents for the  $\alpha$ -random walks slightly increase as the network size  $N$  becomes larger. Here, the case of  $\alpha = 0$  shows the distribution of existing links on a network. Since the stationary probability of incoming at node  $j$  is  $P_j^\infty \propto K_j^{1+\alpha}$  [35], especially at  $\alpha = 0$ , each of the connected links to  $j$  is chosen at random by the probability  $1/K_j$  for the leaving from  $j$ , therefore a walker visit each link at the same number. Figure 2 shows that the frequency of visited links by the  $\alpha$ -random walks at  $\alpha = \pm 1$  is different even for the degrees 3 and 4 in a generalized MSQ network. On the thick lines, a walker tends to visit high population (diagonal) areas colored by orange and red in the case of  $\alpha = 1$ , while it tends to visit low population peripheral (corner) areas in the case of  $\alpha = -1$ . Thus, the case of  $\alpha = 1$  is expected to selectively cover high population areas which has a lot of communication requests in cities. Note that the absolute value of  $\alpha$  should be not too large, since a walker is trapped between high/low degree nodes in a long time as the ping-pong phenomena which does not contribute to the search of targets.

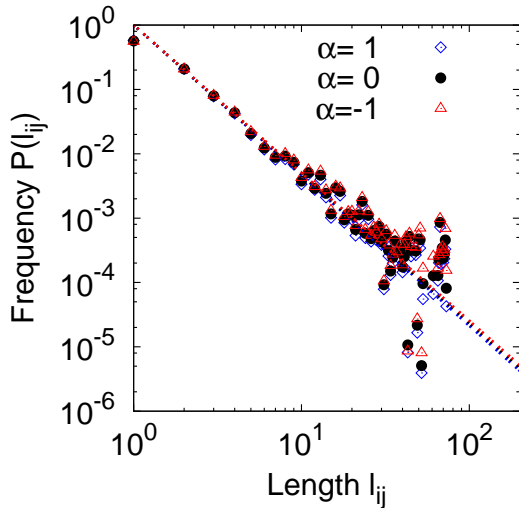


Figure 1. Length distribution of visited links on generalized MSQ networks by an  $\alpha$ -random walker in  $10^6$  time steps. The marks of blue diamond, black circle, and red triangle correspond to the cases of  $\alpha = 1, 0, -1$ , respectively. These results are obtained by the average of 100 networks for  $N = 2000$ .

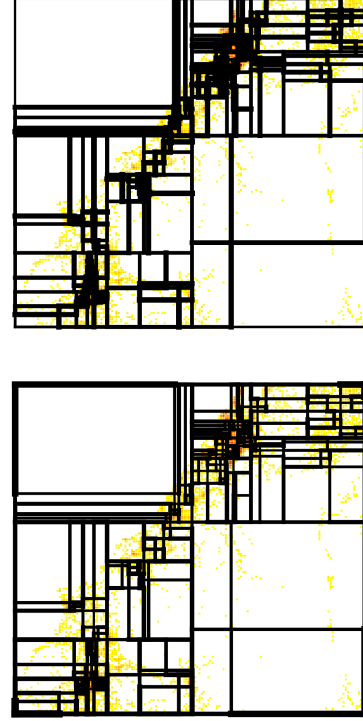


Figure 2. Visualization examples of the visited links by  $\alpha$ -random walks at  $\alpha = 1$  (Top) and  $\alpha = -1$  (Bottom) on a generalized MSQ network for  $N = 500$ . The thickness of link indicates the frequency of visiting in  $10^6$  time steps. From light to dark: white, yellow, and orange to red, the color gradation on a mesh block is proportionally assigned to the population. Many nodes represented as cross points of links concentrate on high population (dark: orange and red) areas on the diagonal direction. In the upper left and lower right of square, corner triangle areas lighted by almost white are the sea of Japan and the Hakusan mountain range.

We investigate the search efficiency for the  $\alpha$ -random walk on a generalized MSQ network, and compare the efficiency with that for Levy flights on a  $L \times L$  square lattice with periodic boundary conditions [12]. As shown in Fig. 3, a walker constantly looks for targets (destination nodes of packets) scanning on a link between two nodes in the generalized MSQ network. If a target exists in the vision area of  $r_v$  hops for the up/down/left/right directions from the center position, a walker gets it and return to the position on the link for continuing the search on the same direction. When more than one targets exit in the area, a walker gets all of them successively in each direction, and return to the position. Only at a node of rectangle, the search direction is changeable along one of the connected links. Thus, the search is restricted on the edges of rectangle in the generalized MSQ network. While the search direction of a Levy flight on the square lattice [12] is selectable from four directions of horizontal and vertical at all times after getting a target in the scanning with the vision area of  $r_v$  hops, moreover, the length of scan follows  $P(l_{ij}) \sim l_{ij}^{-\mu}$ ,

$l_{ij} > r_v$ . We set a target at the position chosen proportionally to the population around a cross point in  $(L + 1)^2$ , for which the population is defined by the average of four values in its contact mesh regions. In particular, we discuss the destructive case [12]: once a target is detected by a walker, then it is removed and a new target is created at a different position chosen with the above probability.

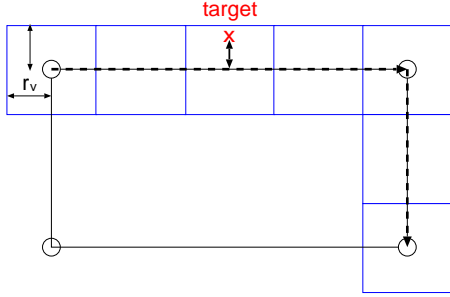


Figure 3. Searching in a generalized MSQ network. Each blue square represents a vision area, and is scanned (from left to right, from top to bottom in this example) by the walker on an edge between two nodes (denoted by circles) of a rectangle. For a target in the area, the walker moves to get it and returns on the link.

The search efficiency [11], [12], [13] is defined by

$$\eta \stackrel{\text{def}}{=} \frac{1}{M} \sum_{m=1}^M \frac{N_s}{L_m}, \quad (1)$$

$$\lambda \stackrel{\text{def}}{=} \frac{(L + 1)^2}{N_t 2r_v}, \quad (2)$$

where  $L_m$  denotes the traversed distance counted by the lattice's unit length until detecting  $N_s = 50$  targets from the total  $N_t$  targets in the  $m$ th run. We consider a variety of  $N_t = 60, 100, 200, 300, 400$ , and  $500$  for investigating the dependency of the search efficiency on the number  $N_t$  of targets. The quantity  $\lambda$  represents the mean interval between two targets for the scaling of efficiency by target density. We set  $M = 10^3$  and  $r_v = 1$  for the convenience of simulation. Intuitively, the sparse and dense structures according to the network size  $N$  have the advantage and disadvantage in order to raise the search efficiency in the generalized MSQ network. Although the scanned areas are limited by some large rectangle holes as  $N$  is small, a walker preferably visits the high population areas which include many targets. While the scanned areas are densely covered as  $N$  is large, the search direction is constrained on long links of a collapse rectangle, therefore it is rather hard for a walker to escape from a local area in which targets are a few.

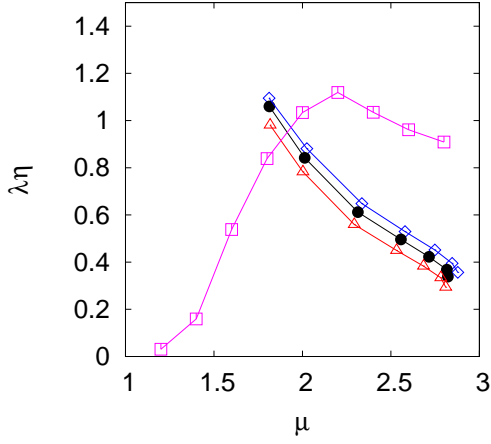
We compare the search efficiency of  $\alpha$ -random walks in the generalized MSQ networks with that of the Levy flights in the square lattice. Figure 4 shows typical trajectories until detecting  $N_s = 50$  targets. On the generalized MSQ network and the square lattice, a walker tends to cover a local area with high population and a wider area, respectively. Without



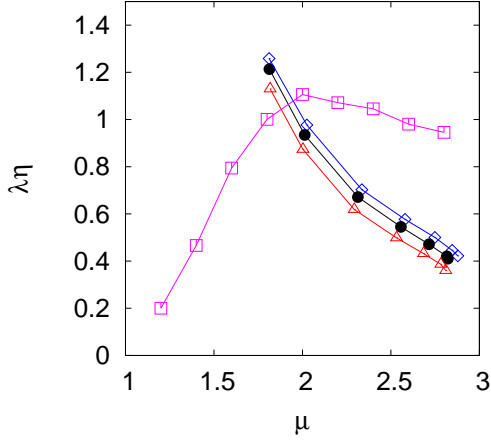
Figure 4. Trajectories of a random walk (Top) at  $\alpha = 0$  on a generalized MSQ network for  $N = 500$  and of a Levy flight (Bottom) for  $\mu = 1.8$  on the square lattice with periodic boundary conditions until detecting  $N_s = 50$  targets in  $N_t = 200$ . Black circle, red circles, and gray rectangle marks denote the start point at the population barycenter, the existing targets, and the removed targets after the detections, respectively. Note that a walker can travel back and forth on a link in the connected path.

wandering peripheral wasteful areas, the generalized MSQ network has a more efficient structure than the square lattice for detecting many targets concentrated on the diagonal areas. Here the exponent  $\mu = 1.8$  of Levy flight corresponds to the slope of  $P(l_{ij})$  in the generalized MSQ network at the optimal size  $N = 500$  for the search efficiency. As shown in Fig. 5(a)(b), the generalized MSQ networks of  $N = 500$  (the diamond, circle, and triangle marks are sticking out at the left) have higher efficiency than the square lattice (the rectangle mark). For the cases with many nodes of  $N \geq 1000$ , the efficiency is decreased more rapidly than that of the Levy flight, however this phenomenon means that many nodes are wasteful and unnecessary to get a high search performance in generalized MSQ networks. When the number  $N_t$  of targets increases in cases from Fig. 5(a) to (b), the curves are shift up, especially for the generalized MSQ networks. The peak value for  $N_t = 200$  is larger than the optimal case of the Levy flight at  $\mu = 2.0$ . Therefore denser targets to that extent around  $N_t = 200$  is suitable, although a case of larger  $N_t > 300$  brings down the search efficiency even for inhomogeneously distributed targets.

In more details, Fig. 6 shows the effect of the number  $N_t$



(a)  $N_t = 100$



(b)  $N_t = 200$

Figure 5. The scaled efficiency  $\lambda\eta$  vs. the exponent  $\mu$ . The marks of blue diamond, black circle, and red triangle correspond to the cases of  $\alpha = 1, 0, -1$ , respectively, in which the increasing values of  $\mu$  are estimated for generalized MSQ networks at  $N = 500, 1000, 2000, 3000, 4000, 5000$ , and 5649:  $N_{max}$  from left to right. The magenta rectangle corresponds to the case of Levy flights on the square lattice. These results are obtained by the average of 100 networks.

of targets on the search efficiency  $\lambda\eta$ . The efficiency firstly increases, then reaches at a peak, and finally decreases for setting more targets. This up-down phenomenon is caused from a trade-off between  $L_m$  and  $N_t$  in Eqs. (1) and (2). Note that the case of size  $N < 500$  is omitted for the generalized MSQ networks. Because sometimes the process for detecting targets until  $N_s$  is not completed, moreover, the variety of link lengths is too little to estimate the exponent as a slope of  $P(l_{ij})$  in the log-log scale. In other words, the estimation is inaccurate because of the short linear part.

#### IV. CONCLUSION

We have considered a scalably self-organized geographical network by iterative divisions of rectangles for load

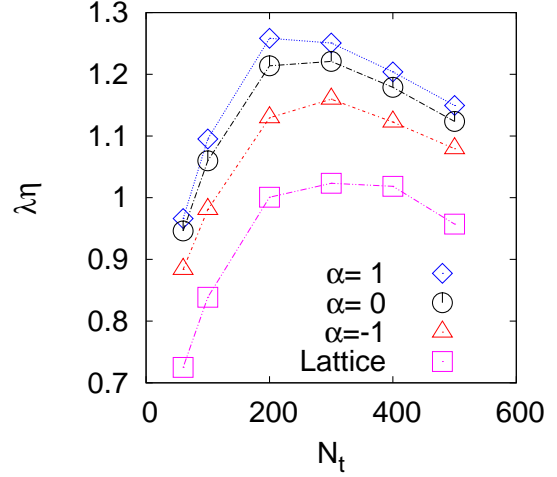


Figure 6. The number  $N_t$  of targets vs. the scaled efficiency  $\lambda\eta$  of  $\alpha$ -random walks on the generalized MSQ networks for  $N = 500$  and of the corresponding Levy flights for  $\mu = 1.8$  (see Fig. 5) on the square lattice. The maximum (optimal) efficiency appears in  $N_t = 200 \sim 300$ . These results are obtained by the average of 100 networks.

balancing of nodes in the adaptive change of their territories according to the increasing of communication requests. In particular, the spatially inhomogeneous distributions of population and the corresponding positions of nodes are important. For the proposed networks, we have investigated the search efficiency in the destructive case [12] with new creations of target after the detections, and shown that the  $\alpha$ -random walks as decentralized routing on the networks have higher search efficiency than the Levy flights known as the optimal strategy [11], [12] on the square lattice with periodic boundary conditions. **One reason for the better performance is the anisotropic covering of high population areas.** Thus, the naturally embedded fractal-like structure is suitable for searching targets in such a realistic situation. In more rigorous discussions about the performance, statistical tests [36] may be useful to clarify the applicability of the proposed method.

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